Higher Education Trajectories

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Freudenthal Institute
Summer School 2022

Universiteit Utrecht

A little bit about me

- I teach at a two-year college in the US.
- I read *China Lectures* early in my doctoral program and my dissertation was influenced by RME.
- I attended Utrecht's 2018 Summer School on Mathematics Education and presented on RME curriculum adaptation.
- My grant team participated in the online summer school teaser in February 2022.

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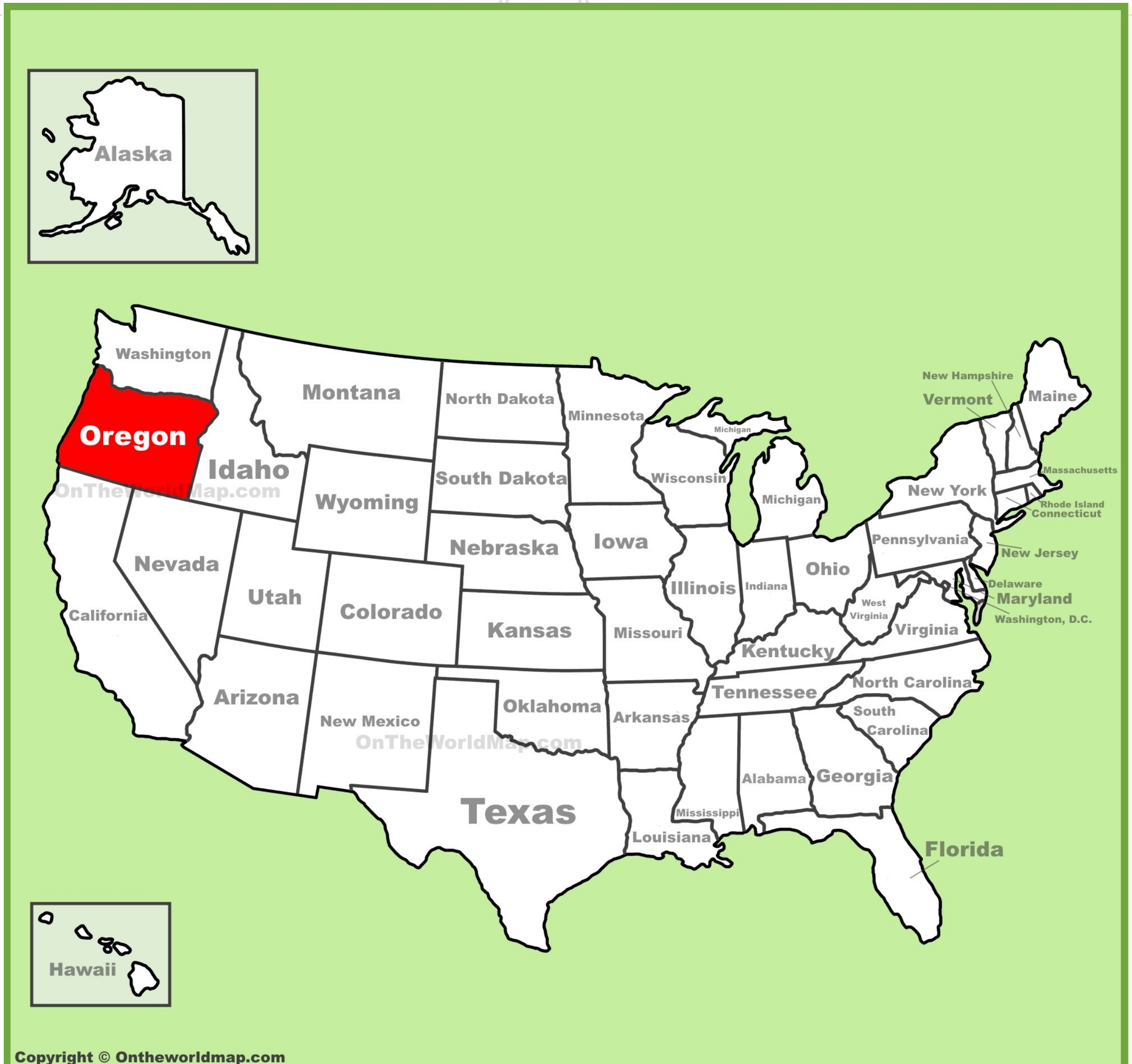
Where do I live and work?



Fun Fact: Utrecht and Portland are “Friendship” Cities

Portland State University is where I earned my PhD

Clackamas Community College is where I teach

Utrecht
25 50 km
25 50 mi
Delfzijl
Leeuwarden
Den Helder
Assen
IJmuiden
Haarlem
AMSTERDAM
Zwolle
Utrecht
Arnhem
Rotterdam
Dordrecht
Nijmegen
Tilburg
Eindhoven
Welkom
UTRECHT SCIENCE PARK
De Uithof
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Universiteit Utrecht
Freudenthal Instituut
Draagt bij aan hoogwaardig onderwijs in wiskunde en natuurwetenschappen via scholing, onderzoek en valorisatie.
Onderzoek
• Bètaonderwijs
• Formeel en informeel leren
• Leermiddelen
• Innovatieve ICT-tools
Onderwijs
Teaching & Learnin
• Ontmoeten
• Inspireren
• Innoveren
Scientific an
www.uu.nl/tll
Hans Freudenthal
1905 - 1990

Background and Overview

- In this workshop, we will engage in some tasks that are **adapted** from a Local Instructional Theory (LIT)* and periodically reflect on our own mathematical activity.

- We will also discuss two key concepts from RME instructional design: **mathematizing** and **guided reinvention**.

*Gravemeijer (2004) used the phrase *local instructional theory* (LIT) "to refer to the description of and rationale for, the envisioned learning route as it relates to a set of instructional activities for a specific topic" (p. 107).

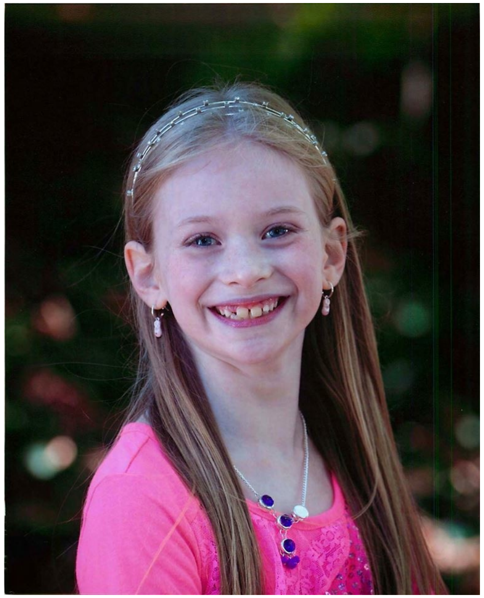
Workshop Plan

- **Wednesday (75 min)**
- Background
- A short story from Primary school
- Task 1
- Reflection
- Task 2

- **Thursday A (75 min)**
- A case of conventionalizing
- **Thursday B (75 min)**
- A little about axiomatizing

An unanticipated sabbatical opportunity

- While I was working on my PhD, I volunteered each Monday for 1.5 hours in my daughter's third grade class.



Third Grade Experience

- The kids sat together in groups, but the teacher only gave the kids worksheets (mechanistic teaching).
- Things were going well until the mid-year exam:
  - Number and Operations (above state average)
  - Data Analysis (above state average)
  - Geometry and Measurement **(well below state average)**
- The researcher: Why was there such a difference in the geometry unit?
- The practitioner: What can we do about it?

Start with the N=1 case (my daughter)

- Examples of test questions:
- “Which angle is **obtuse** (stompzinnig)?”
- “Which triangle is **isosceles** (gelijkbenig)?
- Kyra: “Dad, I don’t know what those words mean.”
- Teacher: “Let’s make flashcards for the kids.”

🤔

- I wanted to design inquiry lessons, but I was not a primary school teacher and I had never taught geometry.

Inspiration came from my wife

- Amy: Can't you use some of that van Hiele stuff from Mike's class?
- Me: Great idea! But wait, those were r ks.

- Amy: Here's your opportunity to use research to inform practice! ☺

Mike Shaughnessy
PSU Professor
NCTM President (2010-2012)

What's My Shape? Burger, W. & Shaughnessy, J. (1986)

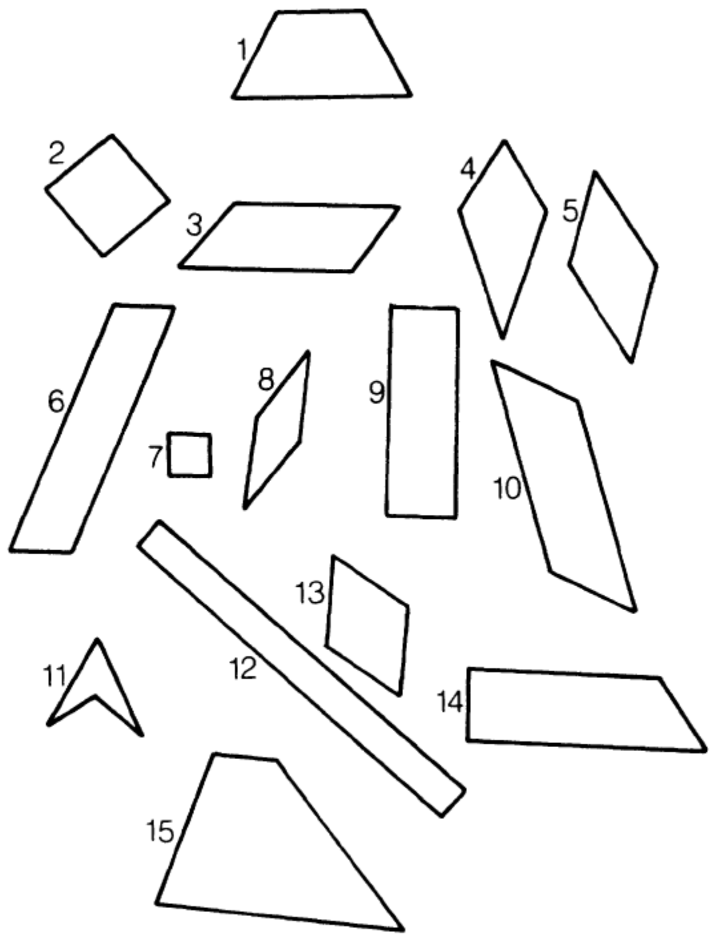


Figure 1. Quadrilaterals to be identified.

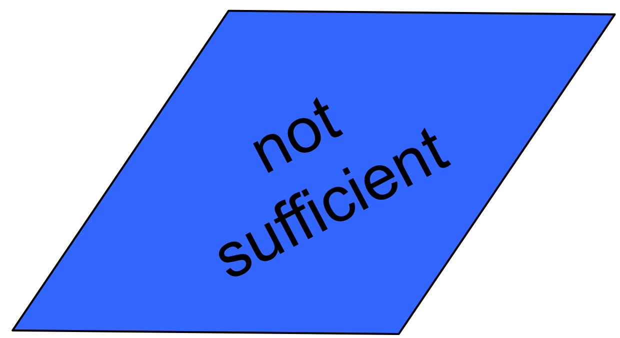
Sorting Quadrilaterals

- **Can you put some of these together that are alike in some way? How are they alike?**

- **Can you put some together that are alike in a different way? How are they alike?**

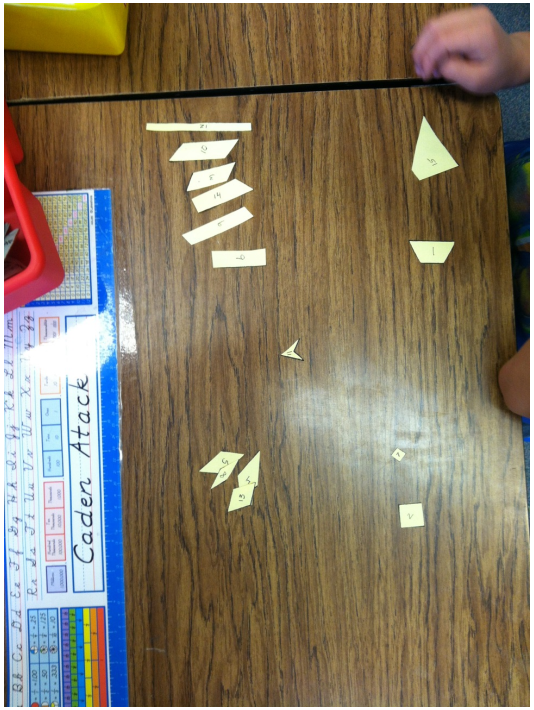
What the research says…

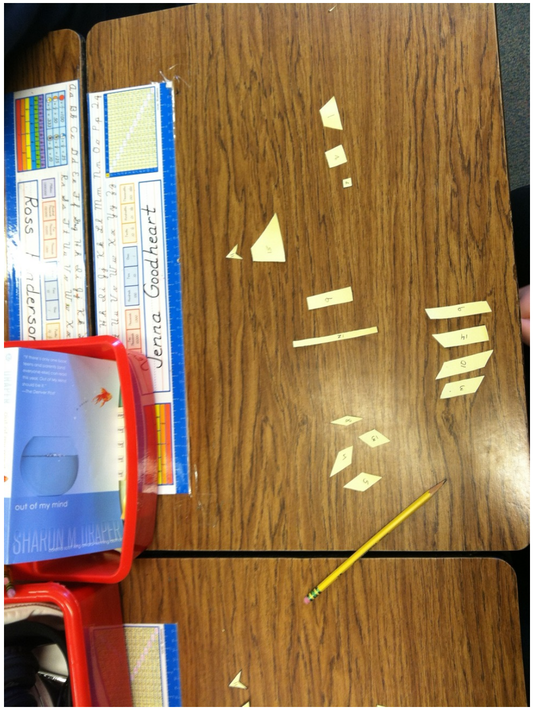
- From a young age, students are often deprived of opportunities to define in a mathematical sense.
- **“A square is a closed figure with four equal sides”**



Sorting Quadrilaterals

- Instead of starting with vocabulary flashcards, we gave them an opportunity to sort and define quadrilaterals.





Lessons learned

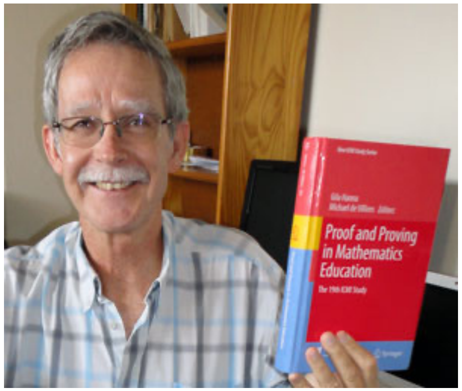
- The students had a lot of fun and did well on the final test in May.
- **Good research tasks make good instructional tasks.**
- These third graders taught me a lot about what students are capable of doing—if you give them the opportunity.

"I don't know if this is right Mr. Yannotta,
but I think a square is both rectangle and a rhombus."

Workshop goals

- **Provide opportunities for you as learners of mathematics to experience and reflect upon some of the design elements of RME**
- Highlight and model some teaching practices for adapting and implementing inquiry-oriented curriculum
- Have fun

RME Inquiry-Oriented Curriculum at the tertiary level



Michael De Villiers
(Boolean Algebra)

Chris Rasmussen
(Differential Equations)

Megan Wawro
(Linear Algebra)

Sean Larsen
(Abstract Algebra)

What is Inquiry-Oriented Instruction (IOI)?

Students in an inquiry-oriented classroom:

"learn new mathematics through inquiry by engaging in mathematical discussions, posing and following up on conjectures, explaining and justifying their thinking, and solving novel problems [27, p. 190]" (Kuster et al., 2017)

Instructors' inquiry into student thinking is a *crucial* aspect of inquiry-oriented instruction

Four IO Instructional Principles:

1. Generating student ways of reasoning
2. Building on student contributions
3. Developing a shared understanding
4. Connecting to standard mathematical language and notation

Students' informal mathematical reasoning is **evoked** and then **leveraged** to develop the more formal mathematics.

Four IO Instructional Principles:

1. Generating student ways of reasoning
2. Building on student contributions
3. Developing a shared understanding
4. Connecting to standard mathematical language and notation

To help ensure that **each student** in the class constructs an understanding of the intended mathematics.

Four IO Instructional Principles:

1. Generating student ways of reasoning
2. Building on student contributions
3. Developing a shared understanding
4. Connecting to standard mathematical language and notation

The development of formal mathematics is still the instructional goal; however, it is a **consequence of** (not a starting point for) students' mathematical reasoning.

The Inquiry-Oriented Curriculum we will explore today

- The Teaching Abstract Algebra for Understanding (TAAFU) curriculum has three parts:
- Group, Isomorphism, & Quotient Groups Units

- The TAAFU curriculum has been used in:

  abstract algebra classes

  mathematics for teachers courses

  transition-to-proof courses

  the mathematics education summer course at Utrecht University.

Active Learning Guiding Principles

Teaching methods and classroom norms that promote:

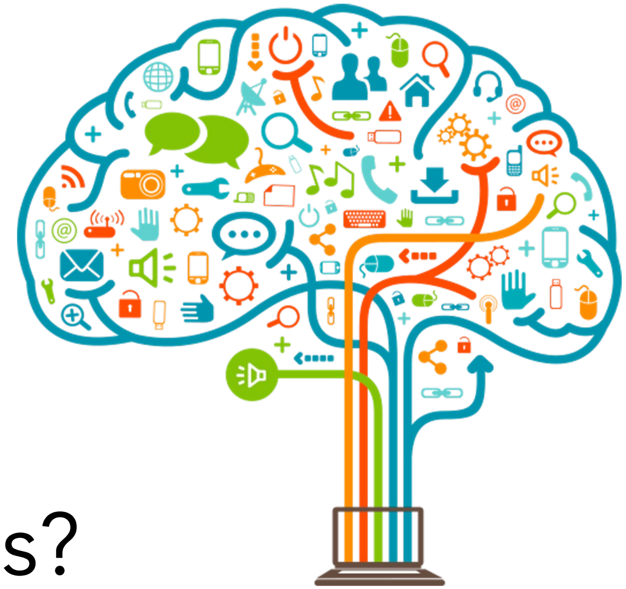
1. Students' deep engagement in mathematical thinking
2. Student-to-student interaction
3. Instructors' interest in and use of student thinking
4. Instructors' attention to equitable and inclusive practices

Source: Laursen, S. L., & Rasmussen, C. (2019). I on the prize: Inquiry approaches in undergraduate mathematics. International Journal of Research in Undergraduate Mathematics Education, 5(1), 129-146.

SEMINAL: Student Engagement in Mathematics through an Institutional Network for Active Learning (collaborative project between San Diego State, University of Colorado-Boulder, and University of Nebraska-Lincoln)

Who is doing the thinking in the classroom?

Students? Or you?



If students, then which ones?
And how often?

Everybody Line Up

Task 1: Counting symmetries

- A **symmetry** is a rigid motion (isometry) that maps a figure to itself.

Experiential introduction:

How many symmetries does an equilateral triangle have?

Task 1: How many symmetries are there for an equilateral triangle?

For each symmetry: 1. Write a verbal description of the symmetry. 2. Draw a diagram to illustrate the symmetry. 3. Create a symbol to represent the symmetry. The symbol should be simple enough to save time writing but should be descriptive as well.

BREAKOUT SPACE:

4 minutes: In your groups, First use the **Once Around Protocol (OAP).** One speaker at a time shares with emphasis that everyone contributes. The only interruptions should be for **clarification.**

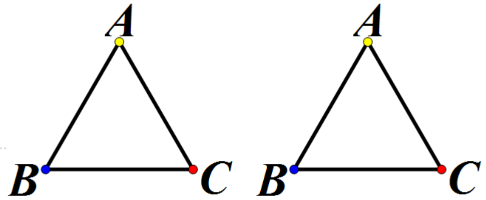
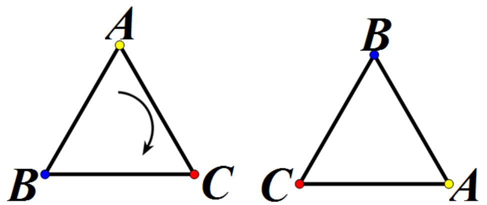
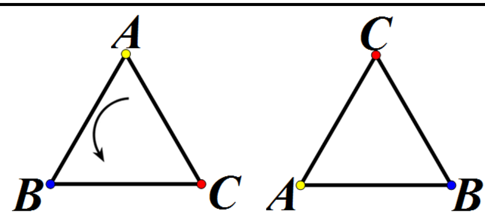
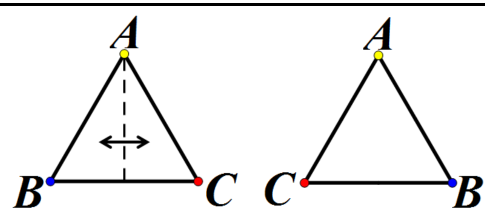
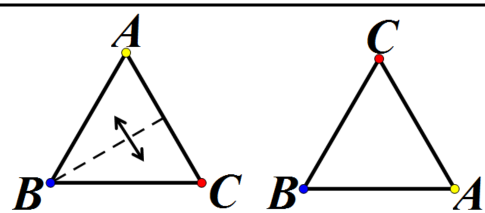
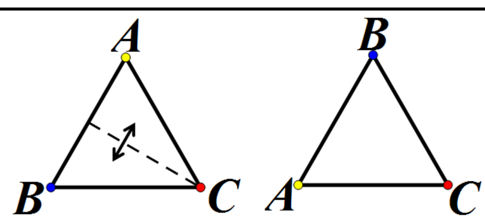
5 minutes: In your groups, engage in **Freeflow Discussion.**

IMPLEMENTATION PLAN:

WHOLE CLASS:

6 minutes: **Private Think Time (PTT)**

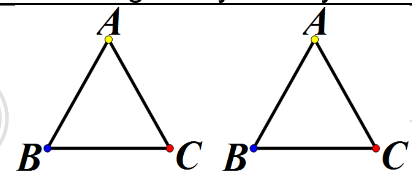
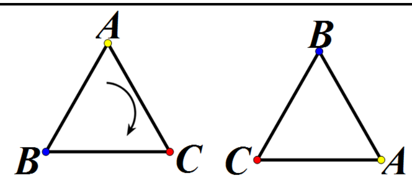
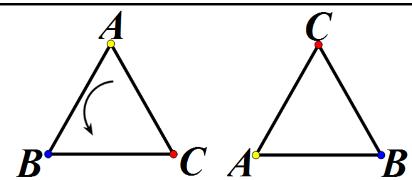
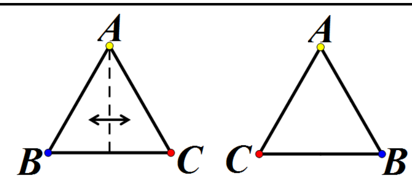
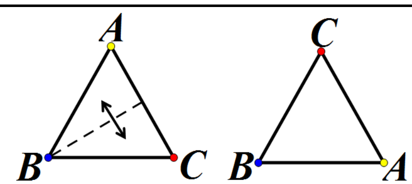
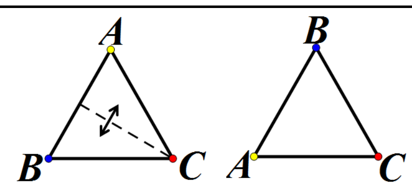
(PTT is quiet time dedicated to the individual engaging with the problem)

Original symmetry	Describe the symmetry
	
	
	
	
	
	

Task 2:

Let $\boldsymbol{R}$ denote a 120° clockwise rotation and let $\boldsymbol{F}$ denote a flip across the vertical axis.

Express each of the six symmetries of an equilateral triangle in terms of $\boldsymbol{F}$'s and $\boldsymbol{R}$'s.

Original symmetry	In terms of ***F*** and ***R***
	
	***R***
	
	***F***
	
	

Breakout 2: How many symmetries of an equilateral triangle

- After students have had opportunities to identify, describe, and symbolize the symmetries of an equilateral triangle at an individual level, we work toward building a common set of symbols that will be used by the class.

IMPLEMENTATION PLAN:

WHOLE CLASS:

2 minutes: Instructor reads directions and models a response or two
1 minute: **Private Think Time**

BREAKOUT SPACE:

6 minutes: In your groups, complete and discuss the remaining entries in the table. Again, you do not need to reach a consensus within your groups.

WHOLE CLASS:

4 minutes: Share some responses in the public space

What happens when we combine two symmetries?

Task 3: Developing a calculus for combining symmetries

- For each combination of two symmetries, figure out which of our six it is equivalent to.
- IMPORTANT: Keep track of any shortcuts or rules that you used.

Reflection of our Day 1 Activities

Day 2: My research-practitioner agenda

- **I am interested in how students construct definitions and create rules and procedures for mathematical systems.**
- **I explore opportunities for advancing mathematical activity in the primary, secondary, and tertiary levels.**

Hans Freudenthal (1905 – 1990) and Realistic Mathematics Education

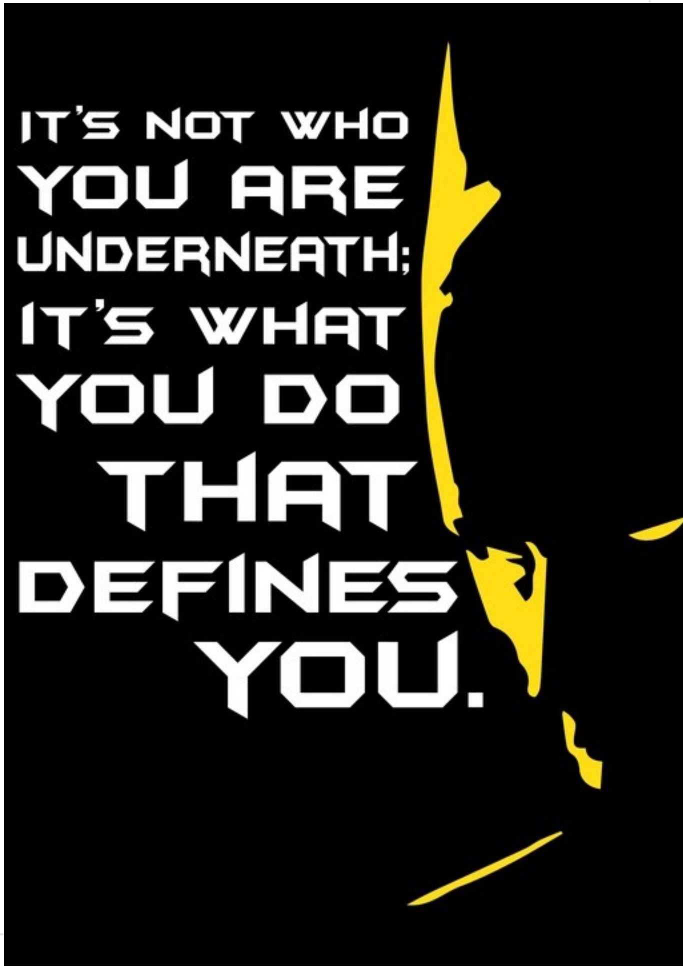
- Born in Germany
- Earned a PhD in Mathematics
- Relocated to the Netherlands prior to 1933
- Founded the academic journal, *Educational studies in mathematics* in 1968 and used it as a platform for RME
- Reconceptualized mathematics from a noun to a verb (mathematizing)
- Adamantly opposed New Math
- Wrote *China Lectures* (1991)

Realistic Mathematics Education

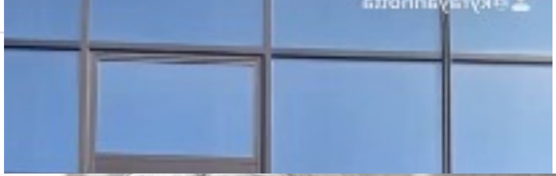
RME places a strong emphasis on two key features:

1) Mathematics is situated as a dynamic human activity.

2) Mathematics must be connected to the reality of everyday life.



Doing mathematics (mathematizing) is something that requires engagement and part of that activity includes operating with and adapting to conventions.



Martial arts & Spanish **Driving & ballet**

If you have the right mindset, you can always learn something new.

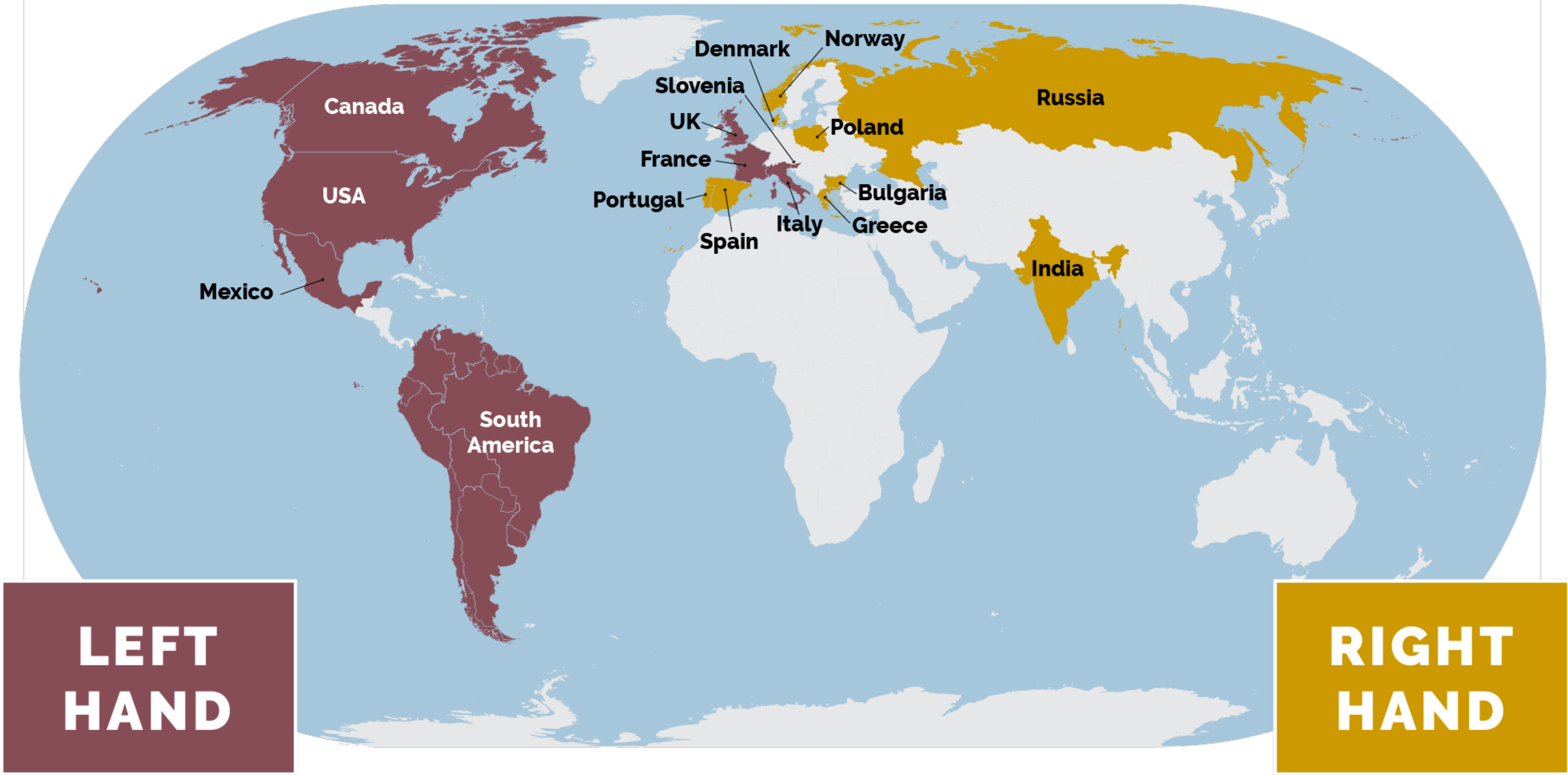
Conventionalizing (Yannotta, 2016)

- Negotiating a set of symbols and representations and establishing procedures for interacting with them

- Conventionalizing is a human activity that addresses a fundamental question:

“How are we going to do things in this context?”

Some Non-mathematical Conventions



On which hand does a wedding ring go?

Some Non-mathematical Conventions

- Right-hand drive (65%) vs. left-hand drive (35%)
- Metric vs. standard measurements
- Have you ever ordered a “large” coffee at Starbucks (corporate)?
- How do you group your students?
- How do you pronounce “inquiry”?

“ɪnˈkwaɪəɹi” (British)

“ˈɪnkwɪɹi” (American)

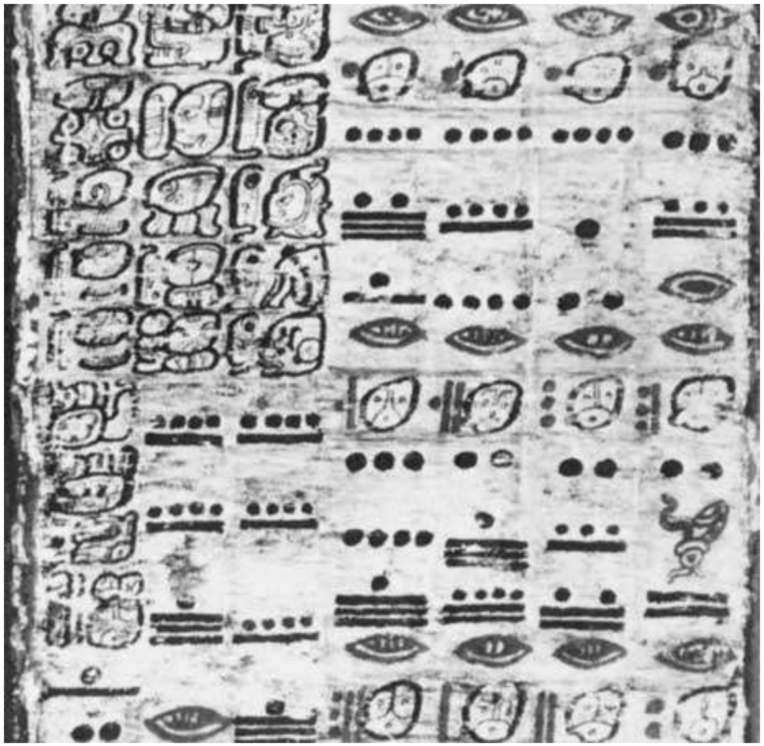
- Is it center or centre?
- K or C (spelling)?

Klaudia

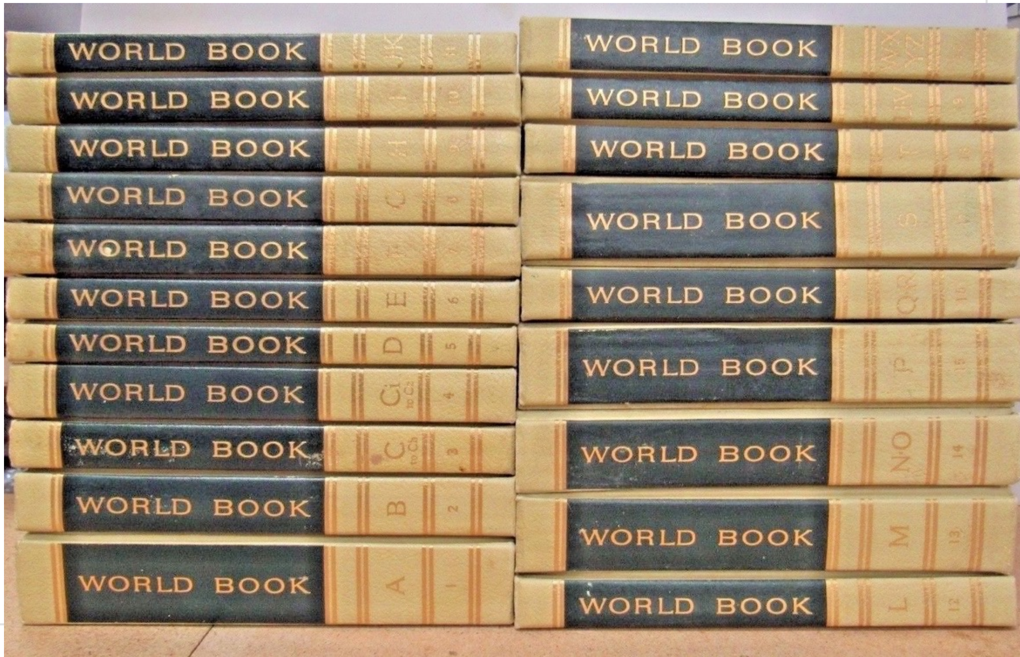
Claudia

My first experience challenging a mathematical convention

- The Dresden Codex

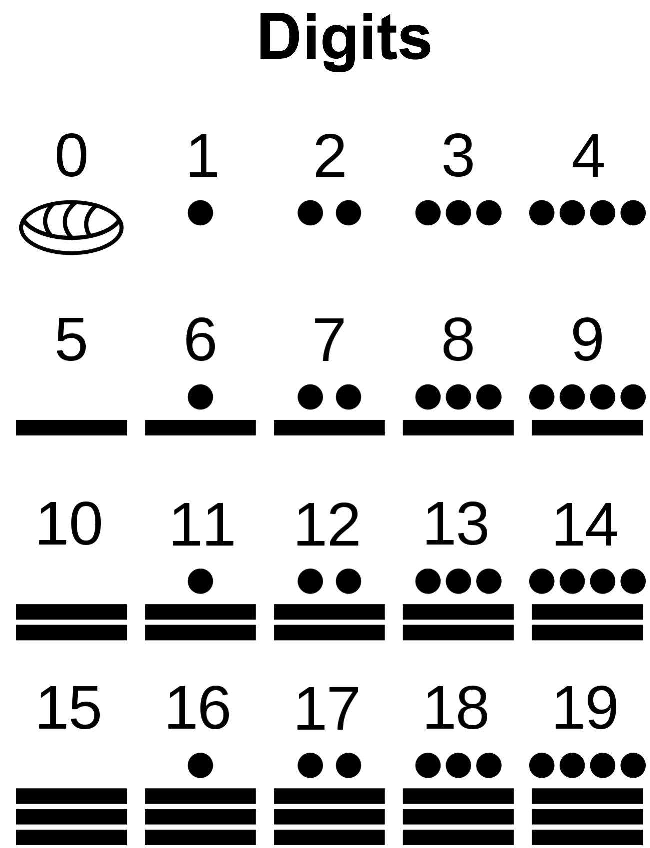


My “Google” 1983 – 1991
World Book Encyclopedia (1966)



The Maya modified vigesimal system

- Numbers are written vertically, with the highest place value at the top.
- In the second position (the twenty's) only the digits 0-17 were used.
- The place values were all based on twenty, except the one in the third position, which is 18*20 instead of 20*20.



Place Values
7200's $18*20^2$
360's $18*20^1$
20's $20 = 20^1$
1's $1 = 20^0$

https://www.dcode.fr/mayan-numbers

Some Mathematical Conventions

- How do we express four and five tenths: 4.5 or 4,5?
- Is (2, 6) a point or an interval? We might consider writing it (2, 6), <2,6>, or even] 2, 6 [.
- Does $\sin^n(x) = (\sin(x))^n$?

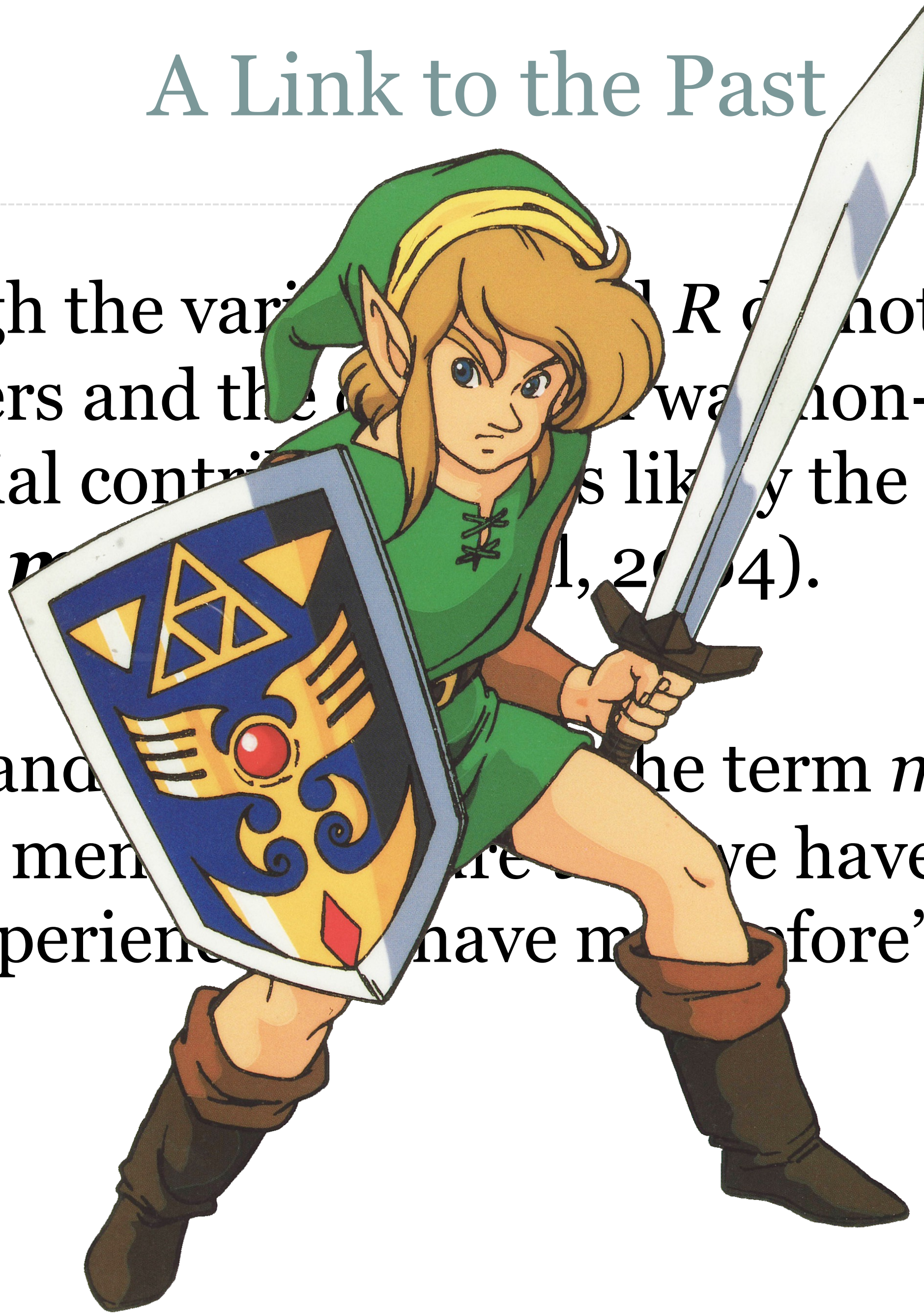
$$\sin^2(x) = (\sin(x))^2 \qquad \sin^{-1}(x) \neq (\sin(x))^{-1}$$

Big take way: Conventions, even in mathematics, are a result of decisions made by human beings and are context dependent.

Ep1: Does $2F+2R = 2(F+R)$?

- The instructor wrote an additive form of two flips followed by two rotations as $2F + 2R$.
- Todd responded, “Well, factor the 2 out of that, right there” and the teacher recorded $2(F+R)=2F+2R$ on the board.

A Link to the Past

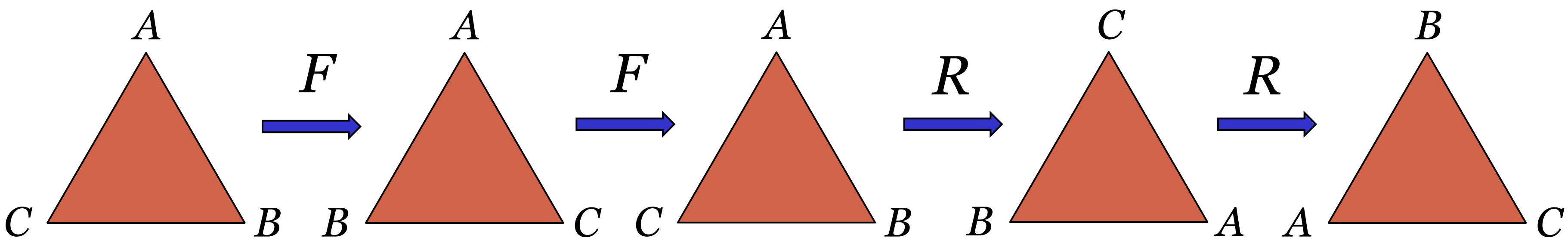


- Even though the vari... ...l *R* d... not represent real numbers and th... ...wa... non-commutative, Todd's initial contri... ...s lik...y the result of him acting on a *m*... ...l, 2...4).

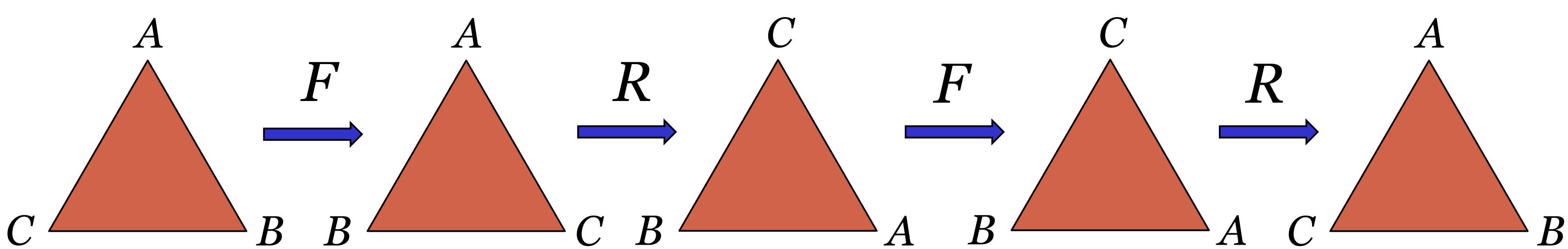
- McGowen and... ...he term *met-before* to describe "a men... ...re... ...ve have *now* as a result of experienc... ...have m... ...efore" (p. 171).

Ep1: Does $2F+2R = 2(F+R)$?

- What does $2F + 2R$ mean?
  - two flips across the vertical axis followed by two 120° clockwise rotations



- What does $2(F+R)$ mean?
  - a flip across the vertical axis followed by a 120° clockwise rotation and then repeat this sequence a second time



Should we use addition or multiplication?

Brian: Have you ever seen a case where addition is going to be different depending on order?

Todd: **Alternating infinite series, it mattered.**

Brian: Have you ever seen a case where something is written as multiplication where order matters ?

Students: **Matrices!**

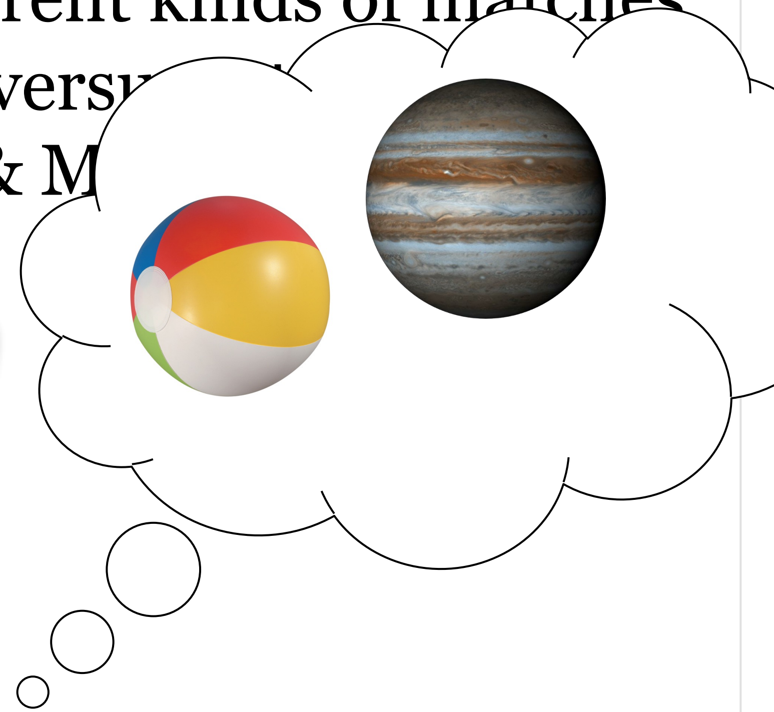
Brian's generative alternative

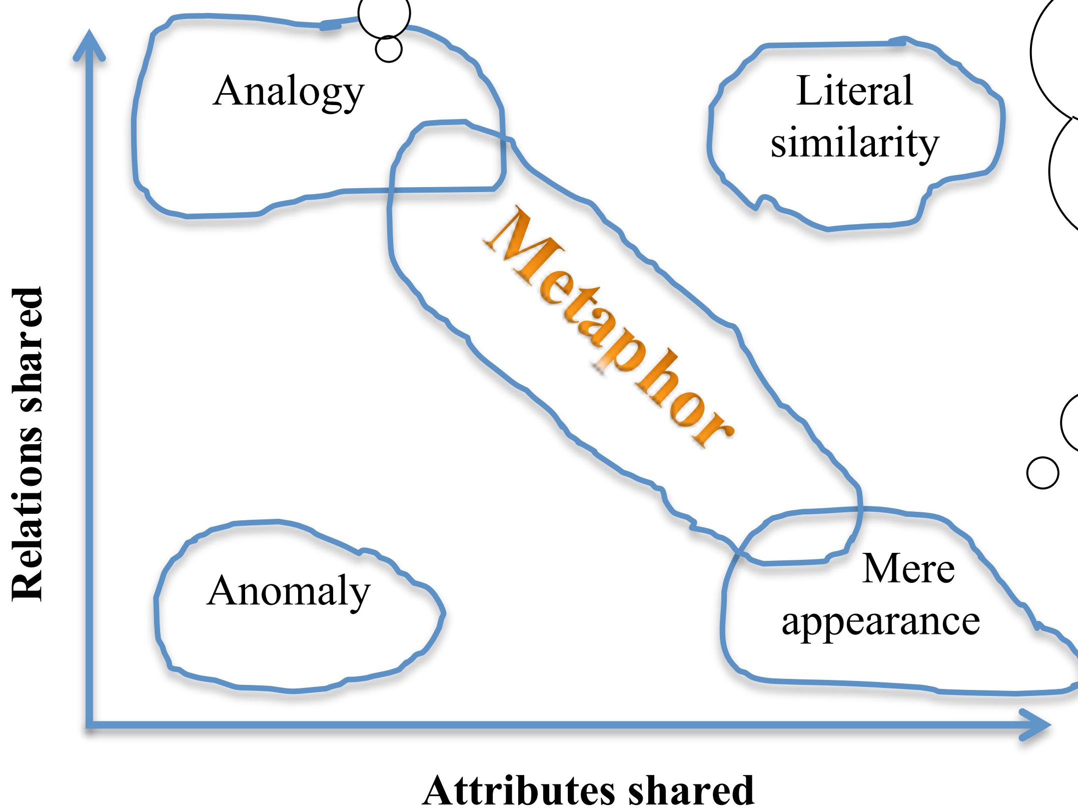
- The use of ***generative alternatives*** supports guided reinvention by allowing the teacher a middle-ground stance between not intervening and assuming the total responsibility of explicating the mathematics within the classroom community (Rasmussen & Marrongelle, 2006).

- The resulting discussion evolved into a series of similarity comparisons (Gentner & Markman, 1997) involving *past* experience.

Comparisons

wing different kinds of matches
gree of relational versu
overlap ad ted from Gentner & M





What was the function of the comparison?

Conventionalizing issue	Addition	Multiplication
Has the operation ever been used in an "order matters" situation?		

Task 3: Completing the operation table

	I	R	R^2	F	FR	RF
I						
R						
R^2						
F						
FR						
RF						

Task 3: Completing the operation table

	I	***R***	***R²***	***F***	***FR***	***RF***
I	*I*	*R*	R^2	*F*	*FR*	*RF*
R	*R*					
R²	R^2					
F	*F*					
FR	*FR*					
RF	*RF*					

Task 3: Completing the operation table

	I	***R***	***R*²**	***F***	***FR***	***RF***
I	*I*	*R*	R^2	*F*	*FR*	*RF*
R	*R*	R^2	*I*			
***R*²**	R^2	*I*	*R*			
F	*F*	*FR*	*RF*			
FR	*FR*	*RF*	*F*			
RF	*RF*	*F*	*FR*			

Task 3: Completing the operation table

	I	**R**	**R^2**	**F**	**FR**	**RF**
I	*I*	*R*	R^2	*F*	*FR*	*RF*
R	*R*	R^2	*I*			
R^2	R^2	*I*	*R*			
F	*F*	*FR*	*RF*	*I*	*R*	R^2
FR	*FR*	*RF*	*F*			
RF	*RF*	*F*	*FR*			

Task 3: Completing the operation table (your turn)

	I	***R***	***R^2***	***F***	***FR***	***RF***
I	*I*	*R*	R^2	*F*	*FR*	*RF*
R	*R*	R^2	*I*			
R^2	R^2	*I*	*R*	*FR*	*RF*	*F*
F	*F*	*FR*	*RF*	*I*	*R*	R^2
FR	*FR*	*RF*	*F*			
RF	*RF*	*F*	*FR*	*R*	R^2	*I*

Task 3: Completing the Operation Table

	I	***R***	***R^2***	***F***	***FR***	***RF***
I	*I*	*R*	R^2	*F*	*FR*	*RF*
R	*R*	R^2	*I*	*RF*	*F*	*FR*
R^2	R^2	*I*	*R*	*FR*	*RF*	*F*
F	*F*	*FR*	*RF*	*I*	*R*	R^2
FR	*FR*	*RF*	*F*	R^2	*I*	*R*
RF	*RF*	*F*	*FR*	*R*	R^2	*I*

Task 4: Identifying Short cuts and Rules

Implementation Notes:

- **Whole Class**: Did you figure all of them out by moving the triangle around or did you use some shortcuts to do the calculations?
- **Teacher:** Share and record a few examples of a rule/short cut.
- **Private Think Time:** What are some rules you used? Make a list.
- **Groups:** Go to your writing spaces and make a list of rules/short cuts.

Task 5: Recompute the table using only the rules

- Implementation notes:
- Work individually for a few minutes (PPT).
- Then, consult with your group to complete the table again using the rules you have recorded.
- If you find you need/want to use a rule that is not in your current list, add it to your list.

Do we need all of these rules?

- Give one example of a rule that could be removed the list.
- How do we justify removing this rule?

Steps in the LIT for reinventing the concept of group (Larsen, 2013)

Step number	Activity
Step 1	Identifying, describing, and symbolizing the symmetries of a specific geometric figure (in this case, an equilateral triangle)
Step 2	Combining pairs of symmetries
Step 3	Developing a calculus for computing combinations of symmetries
Step 4	Axiomatizing the set of rules for computing combinations of symmetries
Step 5	Using the axioms as a model-for reasoning about other contexts
Step 6	Formulating a definition of group

BREAK

Axiomatizing

How does a classroom community engage in collective axiomatizing?

Axiomatization is the search for an adequate definition of a structure (Krygowska, 1971).

Descriptive axiomatizing involves [formulating and] selecting relations in an effort to capture everything that is essential about the structure (Yannotta, 2016b, adapted from De Villiers, 1986).


Anna Zophia Krygowska
1904-1988

Ways to think about axiomatizing (De Villiers, 1986)

- How are the axioms selected?
  - Classical axiomatizing involves selecting self-evident truths and/or those that describe "reality"
  - Modern axiomatizing is born out of logical convenience

    (the selected axioms may or may not be self-evident)

Freudenthal (1973) attributes the distinguishing feature of modern axiomatizing to Hilbert in that it is analogous to chess. The pieces are not defined by their shape, but by the rules they have to obey.

It doesn’t need to look like a pawn, just behave like one

Ways to think about axiomatizing (De Villiers, 1986)

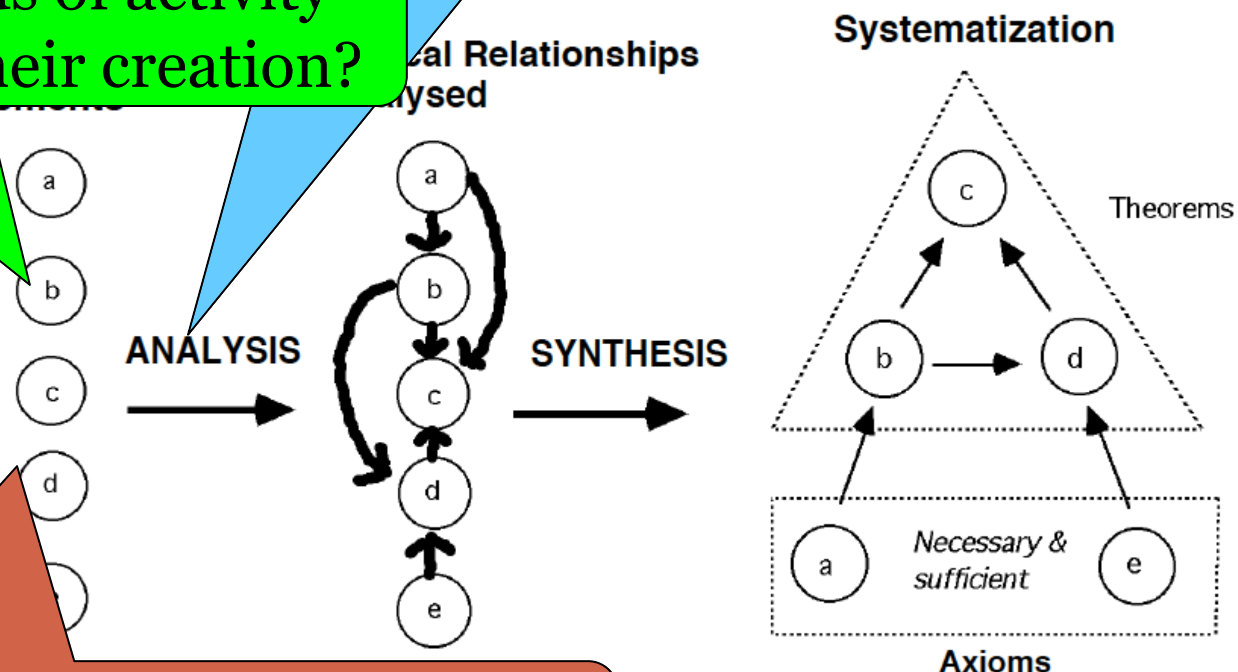
- What function does axiomatizing (eventually) serve?
  - Constructive (a priori) axiomatizing creates new knowledge by omitting/changing/adding axioms within an extant axiomatic system (e.g. the development of non-Euclidean geometries).
  - Descriptive (a posteriori) axiomatizing reorganizes existing knowledge about a given concept by formulating and selecting a subset of essential properties (axioms) to describe the concept. (e.g. Euclid's *Elements* or Hilbert's *Grundlagen der Geometrie*).

De Villiers's (1986) Model for Descriptive Axiomatizing

Most of the collective student activity in my study was consis... of descriptiv... in the Classical sense

How do students analyze these relationships?

What kinds of activity support their creation?



Where do these statements come from in the first place?

Ex. Axiomatic Creation: "Take it to the general case"

Generalize the rule:
$\boldsymbol{RR = R^2}$ **,** $\boldsymbol{R^2R = R^3}$

Kevin's idea:

- $NN = N^2$, $N^2N = N^3$

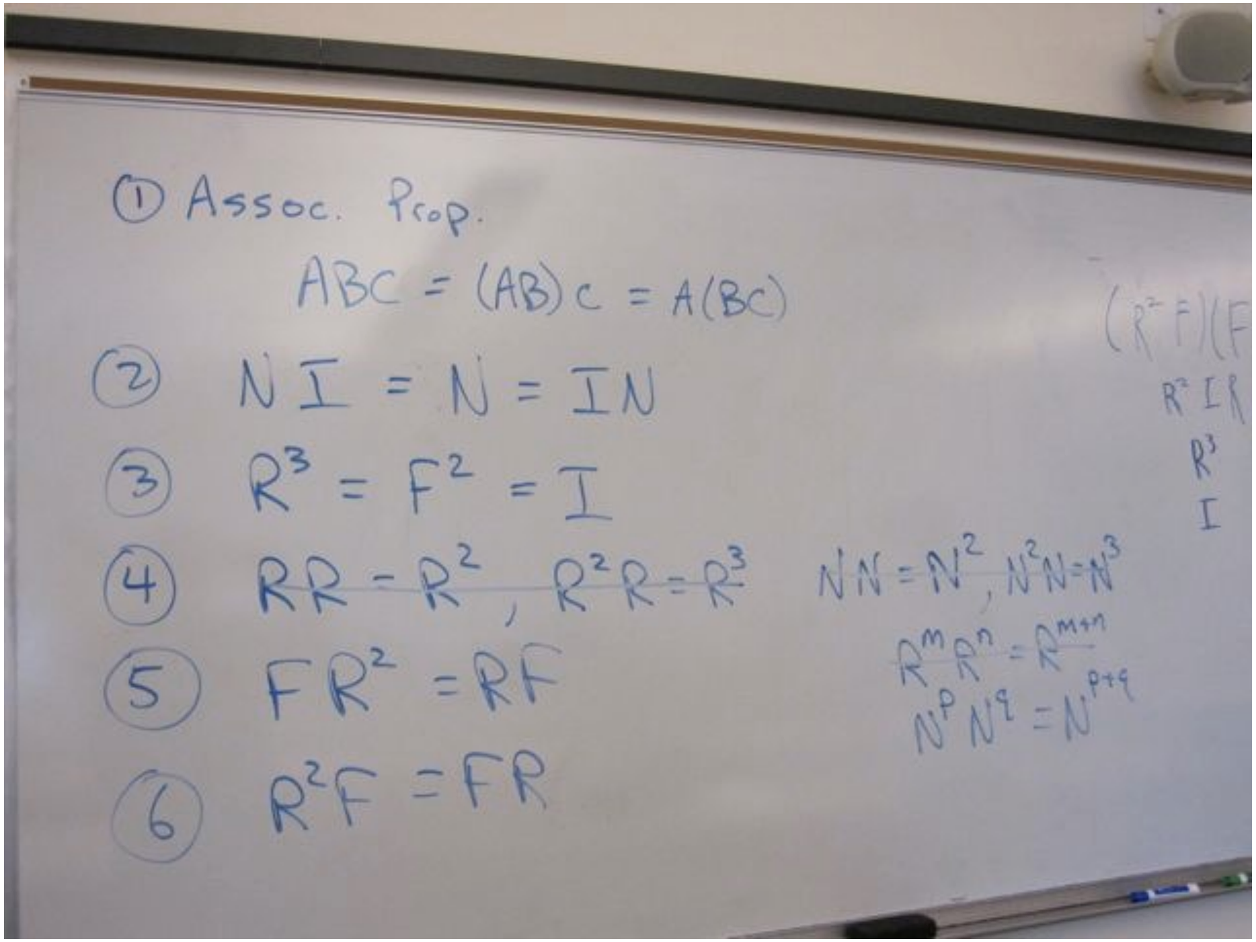
- By changing the variable to N, the rules would apply to both R and F.

Gene's idea:

- $R^{m}R^{n} = R^{m+n}$

- By using variable exponents, we can handle any combination of R's.

Revising the system to include $N^p N^q = N^{p+q}$



Todd's revision incorporates both students' ideas.

The Reflexivity of Conventionalizing and Axiomatizing

	I	R^2	R	F	FR	RF
I	I	R^2	R	F	FR	RF
R	R	I	R^2	RF	F	FR
R^2	R^2	R	I	FR	RF	F
F	F	RF	FR	I	R^2	R^2
FR	FR	F	RF	R^2	I	R
RF	RF	FR	F	R	R^2	I

$FR^2 = RF$

$R^2F = FR$

$R^3 = I$

$F^2 = I$

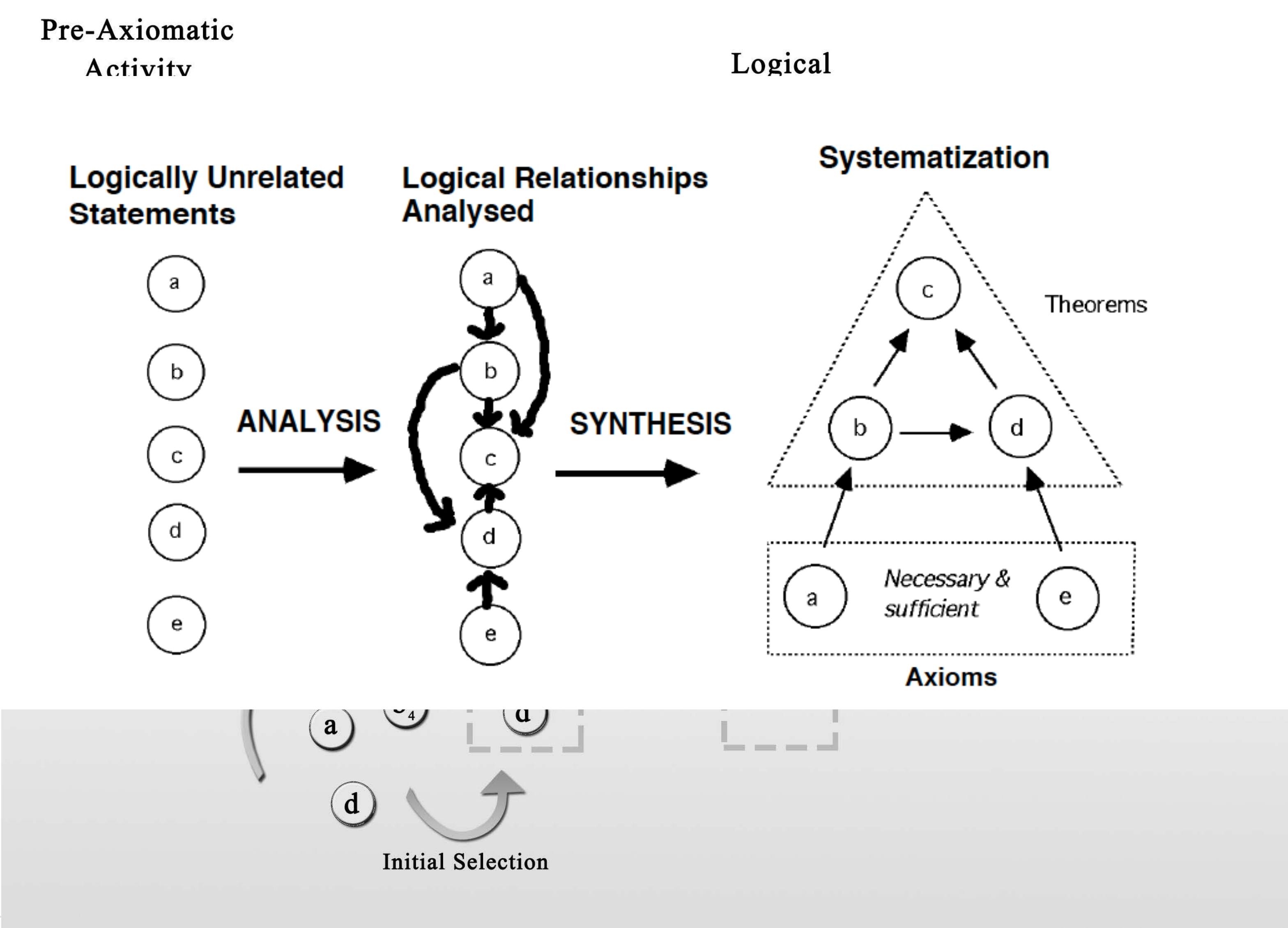
$FRRF = (FR^2)F = RFF = R$

$RFFR = RF^2R = RIR = R^2$

$RFR^2 = R(FR^2) = RRF = R^2F = FR$

This example shows Kevin's symmetry calculations along with some useful relations.

A Revision of De Villiers (1986) Axiomatizing Model



Conclusions

- The exit interview data suggested that most students thought they had changed the way they thought about mathematics.
- The students were empowered and took ownership of the mathematics as they reinvented group.
- RME/inquiry-oriented curriculum can also work for tertiary mathematics.

TAAFU curriculum link: https://taafu.org/old/ioaa/

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